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Branes at orbifolded conifold singularities and supersymmetric gauge field theories

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Abstract: We consider D3 branes at orbifolded conifold singularities which are not quotient singularities. We use toric geometry to study the moduli space of the gauge theories on the D3 branes. We find that topologically distinct phases are related by a flop transition. It is also shown how the conifold singularity is partially solved to an orbifold singularity in some phases if we give expectation values to some of the chiral fields.

Keywords: Brane Dynamics in Gauge Theories, Differential and Algebraic Geometry.

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Contents

1. Introduction

Last years have witnessed great insights into understanding of supersymmetric gauge theory and supergravity theory. We now know that these are complementary descriptions of a single theory on solitonic brane solutions of M theory and string theory. Configurations containing NS fivebranes and D branes in string theory are tools for studying supersymmetric gauge field theory in various dimensions with different supersymmetries (see [1] for a complete set of references up to February 1998).

One of the most important developments is the Maldacena's conjecture which proposes that M or string theory on the $AdS_p \times S^k$, with N units of supergravity k-form field through S^k is dual to a $p-1$ dimensional specific conformal field on the boundary of the AdS_p space [2] (see [3] for an extensive review and a complete set of references). The initial proposal was suited for conformal field theories with maximal supersymmetry, $\mathcal{N} = 4$ in four dimensions. This was obtained by studying D3-branes in flat space. An immediate generalization to D3-branes at orbifold singularities breaks more supersymmetry [4, 5].

Another important class is obtained by studying D3-branes at non-orbifold singularities like conifold singularity. The conifold singularity has been analyzed in [30] where an infrared theory on the worldvolume of D3 branes was proposed. Other results for the case of non-orbifold singularities and their connection to field theories in three and four dimensions have been obtained in [6]–[21].

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In [6, 7], the authors have exploited the fact that the conifold singularity is dual to a system of perpendicular NS5 fivebranes intersecting over a $3+1$ dimensional world-volume. Their result was generalized in [14, 15] for more general conifolds. A duality between D3 branes on these general conifolds and configurations of NS and D4 branes was proposed together with relation between different resolutions of the singularity and displacements of NS branes. In [15], a mirror symmetry was proposed between orbifolded conifolds and generalized conifolds.

In the present paper we consider branes at orbifolded conifolds $\mathcal{C}_{kl}$ which are orbifolds of the three dimensional conifold $xy - uv = 0$ by a discrete group $\mathbb{Z}_k \times \mathbb{Z}_l$. We show that the Higgs branch of the moduli space of the gauge theory is the resolved or (partially) resolved conifold singularity, depending on the values of the FI parameters via toric geometry. We find that topologically distinct phases are related by a flop transition. We also explain the correspondence between D3 branes at orbifolded conifolds and brane configurations obtained after T-dualities. Even though we work on two examples $\mathcal{C}_{2,2}, \mathcal{C}_{2,3}$ of orbifolded conifold $\mathcal{C}_{kl} : xy = z^l$, $uv = z^k$, these examples are general enough to illustrate general phenomena. In [22, 23, 24], D-branes on various other singularities have been studied in the lines of [25] and [14] dealt with generalized conifolds $\mathcal{G}_{kl} : xy = u^k v^l$.

The content of the paper is as follows: in section 2 we give a toric description of the quotient singularity of the three dimensional conifold. In section 3 we review relevant aspects of field theory and brane configurations. We argue why the brane box model is more suitable for orbifolded conifolds than the interval model. In section 4 we derive toric data for the simplest orbifolded conifold $\mathcal{C}_{2,2}$. In section 5 we derive toric data for the orbifolded conifold $\mathcal{C}_{2,3}$. In section 6 we describe different phases of the vacuum moduli space.

2. Toric geometry of orbifolded conifolds

In this section, we will briefly review toric geometry and its physical realization as the moduli space of the D-brane world-volume gauge theory located at the singularity via gauged linear sigma models in order to fix notation. For a detailed review, we refer to [25].

A toric variety is a space which contains algebraic torus $(\mathbb{C}^*)^d$ as an open dense subset. For example, a projective space $\mathbb{P}^d = (\mathbb{C}^{d+1} - \{0\})/\mathbb{C}^*$ is a toric variety because it contains $(\mathbb{C}^*)^d \cong (\mathbb{C}^*)^{d+1}/\mathbb{C}^*$ as an open dense subset. As in the case of the projective space, we will express our toric varieties as a quotient space (this can be thought of as a holomorphic quotient in the sense of the Geometric Invariant Theory [26] or as a symplectic reduction as in gauged linear sigma model. In our cases, these two will be the same [27]). We write a toric variety V_Δ in the form of a

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holomorphic (Geometric Invariant) quotient:

$$V_\Delta = (\mathbb{C}^q - F_\Delta)//(\mathbb{C}^*)^{q-d}\,, \tag{2.1}$$

where q, F_Δ and the action of $(\mathbb{C}^*)^{q-d}$ on $\mathbb{C}^q$ are determined by a combinatorial data Δ. To describe the combinatorial data Δ, let $v_1, \ldots, v_q$ be the vectors in a lattice $\mathbb{N} = \mathbb{Z}^d \subset \mathbb{R} = \mathbb{R}^d$ in general position. Then there are $(q-d)$ independent relations among these vectors:

$$\sum_{i=1}^{q} Q_i^{(a)} v_i = 0\,, \qquad a = 1, \ldots, q-d\,. \tag{2.2}$$

We introduce the corresponding homogeneous coordinates x_i of $\mathbb{C}^q - F_\Delta$ in the holomorphic quotients. In Witten's gauged linear sigma model these coordinates correspond to matter multiplets and $Q^{(a)}$'s correspond to the charges of the matter fields under the $\mathrm{U}(1)^{q-d}$ symmetry which is the maximal compact subgroup of $(\mathbb{C}^*)^{q-d}$. Thus the D-term equations have the form

$$\sum_{i=1}^{q} Q_i^{(a)} |x_i|^2 = r_a\,, \qquad a = 1, \ldots, q-d\,. \tag{2.3}$$

In the holomorphic quotient, the charge matrix whose column vectors consisting of $Q^{(a)}$ determines the action of $(\mathbb{C}^*)^{q-d}$ on $\mathbb{C}^q$ i.e. the action of $(\lambda_1, \lambda_2, \ldots, \lambda_{q-d}) \in (\mathbb{C}^*)^{q-d}$ on $(x_1, \ldots, x_q) \in \mathbb{C}^q$ is given by

$$\left(\lambda_1^{Q_1^{(1)}} \lambda_2^{Q_1^{(2)}} \cdots \lambda_{q-d}^{Q_1^{(q-d)}} x_1, \lambda_1^{Q_2^{(1)}} \lambda_2^{Q_2^{(2)}} \cdots \lambda_{q-d}^{Q_2^{(q-d)}} x_2, \ldots, \lambda_1^{Q_q^{(1)}} \lambda_2^{Q_q^{(2)}} \cdots \lambda_{q-d}^{Q_q^{(q-d)}} x_q\right) \tag{2.4}$$

Here the action can be carried out as written or in two steps, an $(\mathbb{R}_+)^{q-d}$ action followed by an $\mathrm{U}(1)^{q-d}$ action if it is Kähler. This is a point view of gauged linear sigma model approach i.e. symplectic reduction. The quotient depends on the gauge fixing given by the $(\mathbb{R}_+)^{q-d}$ action which is called the moment map in symplectic reduction. In the holomorphic approach, this corresponds to having different choices of F_Δ in (2.1). The different spaces obtained accordingly are birationally equivalent. In a toric diagram this corresponds to different triangulations of a convex cone in the $\mathbb{R}^d$ determined by $\{v_1, \ldots, v_q\}$. The collection of these combinatorial data is denoted by Δ called a fan. We require the quotient space V_Δ to have Gorenstein canonical singularity because V_Δ is needed to be at finite distance on the moduli of Calabi-Yau manifolds. The quotient space V_Δ has Gorenstein canonical singularity if there exists an $u \in \mathbb{Z}^d$ such that $u \cdot v_i = 1$ for all i [28]. Therefore v_i's will lie on the hyperplane with a normal u at a distance $1/\|u\|$ from the origin in $\mathbb{R}^d$. This imposes the following condition on the charge vectors $Q^{(a)}$:

$$\sum_{i=1}^{q} Q_i^{(a)} = 0\,, \qquad a = 1, \ldots, q-d\,, \tag{2.5}$$

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To reformulate our discussions in the language of the gauged linear sigma model more clearly, recall that $\mathbb{C}^q$ is a symplectic manifold with the standard symplectic form $\omega = i\sum_{i=1}^q dz^i \wedge d\bar{z}^{\bar{i}}$. The maximal compact subgroup $G := \mathrm{U}(1)^{q-d}$ of $(\mathbb{C}^*)^{q-d}$ acts covariantly on a symplectic manifold $(\mathbb{C}^q, \omega)$ by symplectomorphisms. The infinitesimal action gives rise to a moment map $\mu : \mathbb{C}^q \to \mathbf{g}^*$ by Poisson brackets. In coordinates, the components of $\mu : \mathbb{C}^q \to \mathbb{R}^{q-d}$ are given by

$$\mu_a = \sum_{i=1}^{q} Q_i^{(a)} |x_i|^2 - r_a\,, \tag{2.6}$$

where r_a are unknown additive constants. The symplectic reduction is then defined as

$$V(r) \equiv \frac{\mu^{-1}(0)}{G}\,. \tag{2.7}$$

The structure of $V(r)$ depends on r. It follows from (2.6) that every $(\mathbb{C}^*)^{q-d}$-orbit in $\mathbb{C}^q$ contributes with at most one point to $V(r)$. The value of r determines the contributing orbits, for a fixed r the set of $(\mathbb{C}^*)^{q-d}$-orbits which do not contribute being precisely F_Δ. The quotient space $V(r)$ carries a symplectic form ω_r by reducing ω to it. The symplectic reduction carries a natural complex structure which allows the reduced symplectic form to become a Kähler form with respect to it.

Now we consider quotient singularities of the conifold (i.e. orbifolded conifold). The conifold is a three dimensional hypersurface singularity in $\mathbb{C}^4$ defined by:

$$\mathcal{C}:\ xy - uv = 0\,. \tag{2.8}$$

The conifold can be realized as a holomorphic quotient of $\mathbb{C}^4$ by the $\mathbb{C}^*$ action given in [29, 30]

$$(A_1, A_2, B_1, B_2) \mapsto (\lambda A_1, \lambda A_2, \lambda^{-1} B_1, \lambda^{-1} B_2) \quad \text{for } \lambda \in \mathbb{C}^*\,. \tag{2.9}$$

Thus the charge matrix is the transpose of $Q' = (1, 1, -1, -1)$ and $\Delta = \sigma$ is a convex polyhedral cone in $\mathbb{N}'_{\mathbb{R}} = \mathbb{R}^3$ generated by $v_1, v_2, v_3, v_4 \in \mathbb{N}' = \mathbb{Z}^3$ where

$$v_1 = (1,0,0)\,, \qquad v_2 = (0,1,0)\,, \qquad v_3 = (0,0,1)\,, \qquad v_4 = (1,1,-1)\,. \tag{2.10}$$

The isomorphism between the conifold $\mathcal{C}$ and the holomorphic quotient is given by

$$x = A_1 B_1\,, \qquad y = A_2 B_2\,, \qquad u = A_1 B_2\,, \qquad v = A_2 B_1\,. \tag{2.11}$$

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We take a further quotient of the conifold $\mathcal{C}$ by a discrete group $\mathbb{Z}_k \times \mathbb{Z}_l$. Here $\mathbb{Z}_k$ acts on A_i, B_j as

$$(A_1, A_2, B_1, B_2) \mapsto \left(e^{-2\pi i/k} A_1, A_2, e^{2\pi i/k} B_1, B_2\right), \tag{2.12}$$

and $\mathbb{Z}_l$ acts as

$$(A_1, A_2, B_1, B_2) \mapsto \left(e^{-2\pi i/l} A_1, A_2, B_1, e^{2\pi i/l} B_2\right). \tag{2.13}$$

Thus they will act on the conifold $\mathcal{C}$ as

$$(x, y, u, v) \mapsto \left(x, y, e^{-2\pi i/k} u, e^{2\pi i/k} v\right) \tag{2.14}$$

and

$$(x, y, u, v) \mapsto \left(e^{-2\pi i/l} x, e^{2\pi i/l} y, u, v\right). \tag{2.15}$$

This quotient is called the orbifolded conifold or the hyper-quotient of the conifold and denoted by $\mathcal{C}_{kl}$. To put the actions (2.11), (2.12) and (2.13) on an equal footing, consider the over-lattice $\mathbb{N}$:

$$\mathbb{N} = \mathbb{N}' + \frac{1}{k}(v_3 - v_1) + \frac{1}{l}(v_4 - v_1). \tag{2.16}$$

Now the lattice points $\sigma \cap \mathbb{N}$ of σ in $\mathbb{N}$ is generated by $(k+1)(l+1)$ lattice points as a semigroup (These lattice points will be referred as a toric diagram). We see that the charge matrix Q is a $(k+1)(l+1)$ by $(k+1)(l+1)-3$ matrix. The discrete group $\mathbb{Z}_k \times \mathbb{Z}_l \cong \mathbb{N}/\mathbb{N}'$ acts on the conifold $\mathbb{C}^4 // \mathrm{U}(1)$ and gives as a quotient the symplectic reduction $\mathbb{C}^{(k+1)(l+1)} // \mathrm{U}(1)^{(k+1)(l+1)-3}$ with the corresponding moment map associated with the charge matrix Q. The new toric diagram for $\mathcal{C}_{kl}$ lies on the plane at a distance $1/\sqrt{3}$ from the origin with a normal vector $(1,1,1)$ and we give a toric diagram on the plane for $\mathcal{C}_{57}$ in figure 1.

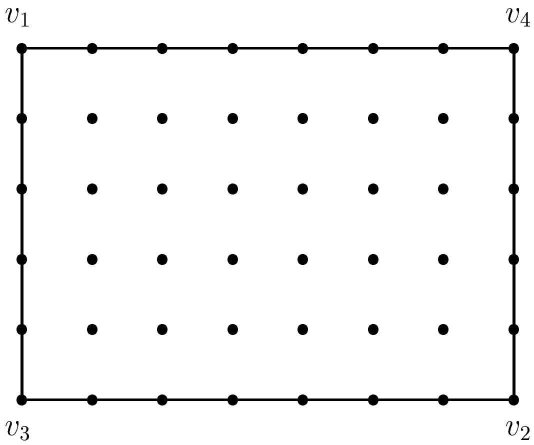

Figure 1: A toric diagram for $\mathbb{Z}_5 \times \mathbb{Z}_7$ hyper-quotient of the conifold, $\mathcal{C}_{57}$.

The action (2.14) and (2.15) of $\mathbb{Z}_k \times \mathbb{Z}_l$ on the conifold $\mathcal{C}$ can be lifted to an action on $\mathbb{C}^4$ whose coordinates are x, y, u, v. The ring of invariants will be $\mathbb{C}[x^l, y^l, xy, u^k, v^k, uv]$ and the orbifolded conifold $\mathcal{C}_{kl}$ will be defined by the ideal $(xy - uv)\mathbb{C}[x^l, y^l, xy, u^k, v^k, uv]$. Thus, after renaming variables, the defining equation for the orbifolded conifold will be

$$\mathcal{C}_{kl} : xy = z^l, \qquad uv = z^k. \tag{2.17}$$

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3. Branes at the singularities and gauge theory

We now put branes to probe the singularity. Consider a system of M D3 branes sitting at the orbifolded conifold $\mathcal{C}_{kl}$ in the transversal direction. In the spirit of [5], the corresponding gauge field theory on the world-volume of $D3$ has been obtained in [6] by investigating the action of the discrete group on the field theory of the conifold developed in [30].

The discrete group $\mathbb{Z}_k \times \mathbb{Z}_l$ acts on the fields A_i, B_i of the conifold theory as in (2.12) and (2.13). By starting with a conifold theory with a group $\mathrm{SU}(klM) \times \mathrm{SU}(klM)$, we obtain via the projection induced by the discrete group action the following $\mathcal{N}=1$ supersymmetric generically chiral gauge theory for a specific choice for the Chan-Paton matrices:

$$\prod_{i=1}^{k}\prod_{j=1}^{l}\mathrm{SU}(M)_{i,j} \times \prod_{i=1}^{k}\prod_{j=1}^{l}\mathrm{SU}(M)'_{i,j} \tag{3.1}$$

with matter content in the following bifundamental representations

Field	**Representations**
$(A_1)_{i+1,j+1;i,j}$	$(\square_{i+1,j+1},\overline{\square}'_{i,j})$
$(A_2)_{i,j;i,j}$	$(\square_{i,j},\overline{\square}'_{i,j})$
$(B_1)_{i,j;i,j+1}$	$(\square'_{i,j},\overline{\square}_{i,j+1})$
$(B_2)_{i,j;i+1,j}$	$(\square'_{i,j},\overline{\square}_{i+1,j})$

as explained in [6]. The superpotential is obtained by substituting the surviving fields into the conifold superpotential:

$$\begin{aligned} W = &\sum_{i,j}(A_1)_{i+1,j+1;i,j}(B_1)_{i,j;i,j+1}(A_2)_{i,j+1;i,j+1}(B_2)_{i,j+1;i+1,j+1} - \\ &-\sum_{i,j}(A_1)_{i+1,j+1;i,j}(B_1)_{i,j;i+1,j}(A_2)_{i+1,j;i+1,j}(B_2)_{i+1,j;i+1,j+1} \end{aligned} \tag{3.2}$$

Moreover, by giving a vev to all the fields $(A_2)_{i,j;i-1,j-1}$, we obtain an $\prod_{i,j}\mathrm{SU}(M)_{i,j}$ gauge theory with surviving chiral multiplets $(A_1)_{i,j;i-1,j-1}, (B_1)_{i,j;i,j+1}, (B_2)_{i,j;i,j+1}$. The superpotential for these fields will be

$$\begin{aligned} W = &\sum_{i,j}(A_1)_{i,j;i-1,j-1}(B_1)_{i-1,j-1;i-1,j}(B_2)_{i-1,j;i,j} - \\ &-(A_1)_{i,j;i-1,j-1}(B_1)_{i-1,j-1;i,j-1}(B_2)_{i,j-1;i,j}\,. \end{aligned} \tag{3.3}$$

This field theory is the one on the world-volume of D3 branes transverse to an orbifold $\mathbb{C}^3/\mathbb{Z}_k \times \mathbb{Z}_l$.

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We now discuss how to arrive from the configurations with D3 branes at conifold singularities to configurations with D4 or D5 branes together with both types of NS branes.

The orbifolded conifold equation

$$\mathcal{C}_{kl} : xy = z^l\,, \qquad uv = z^k \tag{3.4}$$

allows us to consider the orbifolded conifold as a $\mathbb{C}^* \times \mathbb{C}^*$ fibration over the z plane. In other words, for generic values of z, the pairs of variables (x, y) and (u, v) describe $\mathbb{C}^* \times \mathbb{C}^*$. Because we have a $\mathbb{C}^* \times \mathbb{C}^*$ fibration over the z plane, we have two different U(1) orbits, one in each $\mathbb{C}^*$ fiber. So we can perform one T-duality on one of the orbits or two T-dualities along both of these orbits. If we make one T-duality we obtain a configuration with k NS branes on a circle and the final configuration is at a Z_l singularity. As first explained in [31] and developed in [6], this is a chiral theory. Because we still have a singularity which cannot be controlled by removing NS branes, it is more advantageous to do both T-dualities in order to use all the geometrical information. By making the second T-duality, we arrive to a brane box configuration with two compact directions which contains D5 branes together with both types of NS branes. Therefore the geometry allows us to study the Kähler deformation of the orbifolded conifold with brane boxes. As explained in [15], in order to account the number of Kähler structure parameters necessary to completely solve the singularity, we need to modify the intersections of the NS branes. Instead of having perpendicular NS branes intersecting at right angles we have "diamonds" at their intersection. By closing the diamonds we turn off the B field and by rotating the diamonds on a plane perpendicular on the D5 brane we resolve the singularity to $\mathbb{C}^3/\mathbb{Z}_k \times \mathbb{Z}_l$.

4. The orbifolded conifold $\mathcal{C}_{22}$

Consider a system of M D3 branes sitting at the orbifolded conifold

$$\mathcal{C}_{22} : xy = uv = z^2\,. \tag{4.1}$$

As explained before, this is a chiral theory with the gauge group:

$$\prod_{i,j=1}^{2} \mathrm{SU}(M)_{i,j} \times \prod_{i,j=1}^{2} \mathrm{SU}(M)^{'}_{i,j}\,. \tag{4.2}$$

Because the T-dual theory contains NS branes which are perpendicular, the adjoint fields become massive and they are integrated out, leaving only quadratic terms in

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the superpotential. For simplicity we denote the 16 fields by:

$$
\begin{array}{llll}
A_{11} = (A_1)_{22;11}\,, & A_{12} = (A_1)_{21;12}\,, & A_{13} = (A_1)_{12;21}\,, & A_{14} = (A_1)_{11;22}\,, \\
A_{21} = (A_2)_{11;11}\,, & A_{22} = (A_2)_{12;12}\,, & A_{23} = (A_2)_{21;21}\,, & A_{24} = (A_2)_{22;22}\,, \\
B_{11} = (B_1)_{11;12}\,, & B_{12} = (B_1)_{12;11}\,, & B_{13} = (B_1)_{21;22}\,, & B_{14} = (B_1)_{22;21}\,, \\
B_{21} = (B_2)_{11;21}\,, & B_{22} = (B_2)_{12;22}\,, & B_{23} = (B_2)_{21;11}\,, & B_{24} = (B_2)_{22;12}\,.
\end{array}
\tag{4.3}
$$

The D term equations are:

$$
\begin{aligned}
|A_{14}|^2 + |A_{21}|^2 - |B_{12}|^2 - |B_{23}|^2 &= \xi_1\,, \\
|A_{13}|^2 + |A_{22}|^2 - |B_{11}|^2 - |B_{24}|^2 &= \xi_2\,, \\
|A_{12}|^2 + |A_{23}|^2 - |B_{14}|^2 - |B_{21}|^2 &= \xi_3\,, \\
|A_{11}|^2 + |A_{24}|^2 - |B_{13}|^2 - |B_{22}|^2 &= \xi_4\,, \\
|B_{21}|^2 + |B_{11}|^2 - |A_{11}|^2 - |A_{21}|^2 &= \xi_5\,, \\
|B_{22}|^2 + |B_{12}|^2 - |A_{12}|^2 - |A_{22}|^2 &= \xi_6\,, \\
|B_{23}|^2 + |B_{13}|^2 - |A_{13}|^2 - |A_{23}|^2 &= \xi_7\,, \\
|B_{24}|^2 + |B_{14}|^2 - |A_{14}|^2 - |A_{24}|^2 &= \xi_8\,,
\end{aligned}
\tag{4.4}
$$

where the FI parameters satisfy the constraint

$$
\sum_{i=1}^{8} \xi_i = 0\,. \tag{4.5}
$$

The superpotential is

$$
\begin{aligned}
W = {} & A_{11}B_{11}A_{22}B_{22} + A_{12}B_{12}A_{21}B_{21} + A_{13}B_{13}A_{24}B_{24} + A_{14}B_{14}A_{23}B_{23} - \\
& - A_{11}B_{21}A_{23}B_{13} - A_{12}B_{22}A_{24}B_{14} - A_{13}B_{23}A_{21}B_{11} - A_{14}B_{24}A_{22}B_{12}\,.
\end{aligned}
\tag{4.6}
$$

There are 16 F-term constraints obtained by taking the derivative of this superpotential with respect to the 16 fields but not all of them are independent. As opposed to other field theories obtained on D3 branes at singularities, our case involves chiral fields so the F term equations are equalities between two products of three fields For example the relation obtained after taking the derivative with A_{11} is $B_{11}A_{22}B_{22} = B_{21}A_{23}B_{13}$ and the rest of 15 equations are similar. After solving the independent F-term equations, we arrive at 10 independent fields, the rest of 6 fields being expressed in terms of the 10 independent fields. We chose $A_{24}, A_{13}, A_{14}, B_{12}, B_{13}, B_{14}, B_{21}, B_{22}, B_{23}, B_{24}$ as the independent variables. The solution for the F-term equations is given in table 1.

	A_{24}	A_{13}	A_{14}	B_{12}	B_{13}	B_{14}	B_{21}	B_{22}	B_{23}	B_{24}
A_{11}	0	0	1	0	−1	1	−1	0	1	0
A_{12}	0	1	0	0	1	−1	0	−1	0	1
A_{21}	1	0	0	−1	0	1	−1	1	0	0
A_{22}	1	1	−1	−1	1	0	0	0	0	0
A_{23}	1	1	−1	0	1	−1	0	0	−1	1
B_{11}	0	0	0	1	1	−1	1	−1	−1	1

Table 1: The solution for the F-term equations.

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We can now proceed to obtain the vacuum moduli space in the usual way, i.e. by imposing the F-term constraints and the D-term constraints in the form of symplectic quotients as in the gauged linear sigma model. If we impose only the F-term constraints, we can express the 16 fields in terms of the 10 independent fields which gives a cone $\mathbf{M}_+$ in $\mathbf{M} = \mathbb{Z}^{10}$. To construct this as a symplectic quotient, we consider the dual cone $\mathbb{N}_+$ in $\mathbb{N} = \mathrm{Hom}\,(\mathbf{M}, \mathbb{Z})$ and it turns out that the dual cone $\mathbb{N}_+$ is generated by 24 lattice points. Thus we have a map

$$T : \mathbb{Z}^{24} \to \mathbb{N}, \tag{4.7}$$

which is the following 10×24 matrix T

$$\begin{pmatrix}
0&0&1&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&1&0&0&0\\
0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&1&1&1&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&1&0&1&0&0&1&0&0&0&0&0&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&1&1&0&1&0&0&1&0\\
0&0&0&0&0&0&1&1&1&0&0&0&0&1&1&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&1&0&0&0&0&1&1&0&1&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&1&0&1&1&0&1\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&1&1&1\\
0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&1&0\\
1&0&0&1&1&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0
\end{pmatrix}$$

The transpose of the kernel of T is then the 14×24 charge matrix Q

$$\begin{pmatrix}
2&-1&0&-1&-1&1&1&0&-1&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&1\\
1&0&1&0&-1&0&-1&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&1&0\\
1&0&0&-1&-1&1&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&1&0&0\\
1&0&-1&-1&0&1&1&0&-1&0&0&0&-1&0&0&0&0&0&0&0&1&0&0&0\\
1&0&0&-1&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&1&0&0&0&0\\
1&0&-1&-1&0&1&0&0&0&0&0&0&-1&0&0&0&0&0&1&0&0&0&0&0\\
1&-1&1&0&0&-1&0&0&0&0&-1&0&0&0&0&0&0&1&0&0&0&0&0&0\\
1&-1&1&0&-1&0&0&0&0&0&-1&0&0&0&0&0&1&0&0&0&0&0&0&0\\
1&-1&0&0&0&0&0&0&0&0&0&0&-1&0&0&1&0&0&0&0&0&0&0&0\\
1&0&0&0&-1&0&0&0&-1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&-1&-1&0&0&0&-1&0&0&1&0&0&0&0&0&0&0&0&0&0\\
1&0&-1&-1&0&0&1&0&-1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&-1&0&-1&0&0&1&0&-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&-1&-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

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Thus we have an exact sequence:

$$0 \longrightarrow \mathbb{Z}^{14} \xrightarrow{{}^tQ} \mathbb{Z}^{24} \xrightarrow{T} \mathbb{N} \longrightarrow 0\,. \tag{4.8}$$

From this sequence, one can see that the moduli space $\mathcal{M}_F$ can be expressed as a holomorphic quotient of $\mathbb{C}^{24}$ by $(\mathbb{C}^*)^{14}$ whose action is specified by Q (or a symplectic quotient by $\mathrm{U}(1)^{14}$) via the map induced by T. To incooperate the D-term constraints, we need to see how the action of $(\mathbb{C}^*)^{10}$ on the toric variety $\mathcal{M}_F$ is represented in these terms. Since the action of $(\mathbb{C}^*)^{10}$ on the open subset $(\mathbb{C}^*)^{10} \subset \mathcal{M}_F$ must be the obvious multiplication, the action of $(\mathbb{C}^*)^{10}$ on $\mathbb{C}^{24}$ is specified the transpose of a 10×24 matrix U such that

$$T \cdot {}^tU = \ \mathrm{Id}_k\,. \tag{4.9}$$

The 10×24 matrix U is

$$\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

The D-term equations are represented by the 7×10 matrix V

$$\begin{pmatrix}
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

We ignored the charges on the dependent fields because they are already encoded in Q. Thus on $\mathbb{C}^{24}$, the D-term constraints are represented by the charge matrix VU.

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Finally, the full set of charges is given by the 21×24 charge matrix $\tilde{Q}$

$$
\begin{pmatrix}
2 & -1 & 0 & -1 & -1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

by concatenating Q and VU. The cokernel of its transpose gives toric data for our vacuum moduli space, denoted by $\mathcal{M}$. After eliminating redundant variables, it is given in the form of a map $T_{\mathcal{M}} : \mathbb{Z}^9 \to \mathbb{Z}^3$:

$$
T_{\mathcal{M}} = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \tag{4.10}
$$

Note that we need to impose conditions on the FI terms in order to eliminate some variables even though the resulting moduli will not depend on the choice of redundant variables in general. The conditions for the FI terms corresponds to the non-vanishing of the redundant homogeneous coordinates [22]. The lattice points given by $T_{\mathcal{M}}$ lie on the plane with a normal vector $(1,1,1)$ at a distance $1/\sqrt{3}$ from the origin. We depict these lattice points

$$
\begin{array}{lll}
v_1 = (2,0,-1)\,, & v_2 = (1,1,-1)\,, & v_3 = (0,2,-1)\,, \\
v_4 = (1,0,0)\,, & v_5 = (0,1,0)\,, & v_6 = (-1,2,0)\,, \\
v_7 = (0,0,1)\,, & v_8 = (-1,1,1)\,, & v_9 = (-2,2,1)
\end{array} \tag{4.11}
$$

in the planar diagram (figure 2). This is exactly a toric diagram for the orbifolded conifold $\mathcal{C}_{22} : xy = uv = z^2$. The corresponding charge matrix $Q_{\mathcal{M}}$ for the toric data

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$T_{\mathcal{M}}$ with the Fayet-Iliopoulos D-term parameters from (4.4) is as follows:

$$Q_{\mathcal{M}} = \begin{pmatrix} 0 & 0 & 0 & 2 & -2 & 0 & -1 & 0 & 1 & 2\xi_1+\xi_2+\xi_4+\xi_5-\xi_7 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & \xi_1-\xi_7 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & \xi_1-\xi_3-\xi_6-\xi_7 \\ 1 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & -\xi_1-\xi_2-\xi_5-\xi_6 \\ 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 & -\xi_2-\xi_3-\xi_6-\xi_7 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & -\xi_2-\xi_6 \end{pmatrix} \tag{4.12}$$

For this choice of redundant variables, the Fayet-Iliopoulos D-term parameters must satisfy

$$\begin{aligned} \xi_1 > 0\,, \qquad & \xi_4 > 0\,, \\ -\xi_6 > 0\,, \qquad & -\xi_7 > 0\,, \\ -\xi_3 - \xi_6 - \xi_7 &> 0\,, \\ -\xi_2 - \xi_6 - \xi_7 &> 0\,, \\ -\xi_2 - \xi_3 - \xi_6 - \xi_7 &> 0\,, \\ -\xi_1 - \xi_2 - \xi_5 - \xi_6 &> 0\,, \\ -\xi_1 - \xi_2 - \xi_3 - \xi_5 - \xi_6 - \xi_7 &> 0\,, \\ \xi_1 + \xi_4 + \xi_5 &> 0\,, \\ \xi_1 + \xi_2 + \xi_4 + \xi_5 &> 0\,. \end{aligned} \tag{4.13}$$

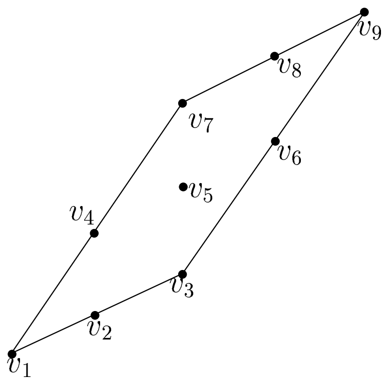

Figure 2: $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of the conifold $\mathcal{C}_{22}$.

5. The orbifolded conifold $xy = z^2, uv = z^3$

In this case we start with a system of D branes sitting at the orbifold conifold singularity

$$\mathcal{C}_{23} : xy = z^2, uv = z^3\,. \tag{5.1}$$

By putting on the M D3 branes on $\mathcal{C}_{23}$, we obtain the field theory with the gauge group:

$$\prod_{i=1}^{2}\prod_{j=1}^{3} \mathrm{SU}(M)_{i,j} \times \prod_{i=1}^{2}\prod_{j=1}^{3} \mathrm{SU}(M)^{'}_{i,j} \tag{5.2}$$

The matter content for this theory is similar to the one encountered for the previous orbifolded conifold but we have 24 fields now instead of 16 as before. For simplicity we denote the 24 fields by:

$$\begin{array}{llll} A_{11} = (A_1)_{22;11}\,, & A_{12} = (A_1)_{21;12}\,, & A_{13} = (A_1)_{32;21}\,, & A_{14} = (A_1)_{31;22}\,, \\ A_{15} = (A_1)_{12;31}\,, & A_{16} = (A_1)_{11;32}\,, & A_{21} = (A_2)_{11;11}\,, & A_{22} = (A_2)_{12;12}\,, \\ A_{23} = (A_2)_{21;21}\,, & A_{24} = (A_2)_{22;22}\,, & A_{25} = (A_2)_{31;31}\,, & A_{26} = (A_2)_{32;32}\,, \\ B_{11} = (B_1)_{11;12}\,, & B_{12} = (B_1)_{12;11}\,, & B_{13} = (B_1)_{21;22}\,, & B_{14} = (B_1)_{22;21}\,, \\ B_{15} = (B_1)_{31;32}\,, & B_{16} = (B_1)_{32:31}\,, & B_{21} = (B_2)_{11;21}\,, & B_{22} = (B_2)_{12;22}\,, \\ B_{23} = (B_2)_{21;31}\,, & B_{24} = (B_2)_{22;32}\,, & B_{25} = (B_2)_{31;11}\,, & B_{26} = (B_2)_{32;12}\,. \end{array} \tag{5.3}$$

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The superpotential is then:

$$\begin{aligned} W = {} & A_{11}B_{11}A_{22}B_{22} + A_{12}B_{12}A_{21}B_{21} + A_{13}B_{13}A_{24}B_{24} + A_{14}B_{14}A_{23}B_{23} + \\ & + A_{15}B_{15}A_{26}B_{26} + A_{16}B_{16}A_{25}B_{25} - A_{11}B_{21}A_{23}B_{13} - A_{12}B_{22}A_{24}B_{14} - \\ & - A_{13}B_{23}A_{25}B_{15} - A_{14}B_{24}A_{26}B_{16} - -A_{15}B_{25}A_{21}B_{11} - A_{16}B_{26}A_{22}B_{12}\,. \end{aligned} \tag{5.4}$$

There are 24 F-term constraints derived from this superpotential, not all of them independent and by solving them we arrive at 14 independent fields, the rest of 10 fields being expressed in terms of these. We choose

$$A_{16},\ A_{26},\ B_{11},\ B_{12},\ B_{13},\ B_{14},\ B_{15},\ B_{16},\ B_{21},\ B_{22},\ B_{23},\ B_{24},\ B_{25},\ B_{26} \tag{5.5}$$

as the independent fields.

The D term equations are:

$$\begin{aligned} |A_{16}|^2 + |A_{21}|^2 - |B_{12}|^2 - |B_{25}|^2 &= \xi_1\,, \\ |A_{15}|^2 + |A_{22}|^2 - |B_{11}|^2 - |B_{26}|^2 &= \xi_2\,, \\ |A_{12}|^2 + |A_{23}|^2 - |B_{14}|^2 - |B_{21}|^2 &= \xi_3\,, \\ |A_{11}|^2 + |A_{24}|^2 - |B_{13}|^2 - |B_{22}|^2 &= \xi_4\,, \\ |A_{14}|^2 + |A_{26}|^2 - |B_{16}|^2 - |B_{23}|^2 &= \xi_5\,, \\ |A_{13}|^2 + |A_{25}|^2 - |B_{15}|^2 - |B_{24}|^2 &= \xi_6\,, \\ |B_{21}|^2 + |B_{11}|^2 - |A_{11}|^2 - |A_{21}|^2 &= \xi_7\,, \\ |B_{22}|^2 + |B_{12}|^2 - |A_{12}|^2 - |A_{22}|^2 &= \xi_8\,, \\ |B_{23}|^2 + |B_{13}|^2 - |A_{13}|^2 - |A_{23}|^2 &= \xi_9\,, \\ |B_{24}|^2 + |B_{14}|^2 - |A_{14}|^2 - |A_{24}|^2 &= \xi_{10}\,, \\ |B_{25}|^2 + |B_{15}|^2 - |A_{15}|^2 - |A_{25}|^2 &= \xi_{11}\,, \\ |B_{26}|^2 + |B_{16}|^2 - |A_{16}|^2 - |A_{26}|^2 &= \xi_{12}\,, \end{aligned} \tag{5.6}$$

where the FI parameters satisfy the constraint

$$\sum_{i=1}^{12} \xi_i = 0\,. \tag{5.7}$$

We want to implement the same procedure as in the previous $\mathbb{Z}_2 \times \mathbb{Z}_3$ orbifolded conifold $\mathcal{C}_{22}$. As before, we can identify the moduli space $\mathcal{M}_F$ of 24 fields satisfying the F-term constraints as a cone $\mathbf{M}_+$ in $\mathbf{M} = \mathbb{Z}^{14}$. The dual cone $\mathbb{N}_+$ is generated by 80 lattice points represented by T. Thus $\mathcal{M}_F$ can be expressed as a symplectic quotient $\mathbb{C}^{80} // \,\mathrm{U}(1)^{66}$ whose action is specified by Q. This can be expressed as the following exact sequence:

$$0 \longrightarrow \mathbb{Z}^{66} \xrightarrow{{}^tQ} \mathbb{Z}^{80} \xrightarrow{T} \mathbb{N} \longrightarrow 0\,. \tag{5.8}$$

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By further imposing the 11 D-term equations from (5.6), we obtain toric data for the vacuum moduli space $\mathcal{M}$ as a three dimensional toric variety $\mathbb{C}^{80}//\,\mathrm{U}(1)^{77}$. Because of huge sizes of the matrices involved, we only write the final toric data after eliminating redundant variables. It is given in the form of a map $T_{\mathcal{M}} : \mathbb{Z}^{12} \to \mathbb{Z}^{3}$:

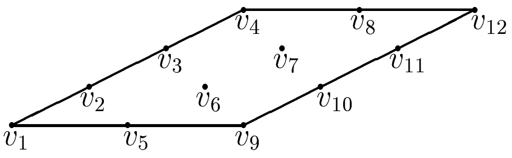

Figure 3: $\mathbb{Z}_3 \times \mathbb{Z}_2$ orbifold of the conifold $\mathcal{C}_{23}$.

$$T_{\mathcal{M}} = \begin{pmatrix} 0 & 1 & 2 & 3 & -1 & 0 & 1 & 2 & -1 & 0 & 1 & -2 \\ 0 & -1 & -2 & -3 & 2 & 1 & 0 & -1 & 3 & 2 & 1 & 4 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{pmatrix} \tag{5.9}$$

The lattice points given by the matrix $T_{\mathcal{M}}$ are as follows:

$$\begin{array}{llll} v_1 = (0,0,1)\,, & v_2 = (1,-1,1)\,, & v_3 = (2,-2,1)\,, & v_4 = (3,-3,1)\,, \\ v_5 = (-1,2,0)\,, & v_6 = (0,1,0)\,, & v_7 = (1,0,0)\,, & v_8 = (2,-1,0)\,, \\ v_9 = (-1,3,-1)\,, & v_{10} = (0,2,-1)\,, & v_{11} = (1,1,-1)\,, & v_{12} = (-2,4,-1)\,, \end{array} \tag{5.10}$$

which are drawn in figure 3. This is exactly the toric data for the $\mathbb{Z}_3 \times \mathbb{Z}_2$ orbifolded conifold $\mathcal{C}_{32}$. The corresponding charge matrix is given by

$$Q_{\mathcal{M}} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -4 & 1 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -3 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \tag{5.11}$$

The Fayet-Iliopoulos D-term parameters corresponding to each row are given by

$$\begin{pmatrix} -3\xi_1 + \xi_3 - 2\xi_4 - \xi_5 + 2\xi_6 - 3\xi_7 + \xi_9 - 2\xi_{10} - \xi_{11} \\ 3\xi_1 - \xi_3 + 2\xi_4 + \xi_5 - 2\xi_6 + 2\xi_7 - \xi_8 - 2\xi_9 + \xi_{10} \\ -3\xi_1 - \xi_2 - 2\xi_4 - \xi_5 + \xi_6 - 3\xi_7 - \xi_8 - 2\xi_{10} - \xi_{11} \\ 2\xi_1 - \xi_3 + \xi_4 - 2\xi_6 + \xi_7 - \xi_8 - 2\xi_9 \\ \xi_1 - \xi_2 - \xi_3 - 2\xi_6 - \xi_8 - 2\xi_9 - \xi_{11} \\ -2\xi_1 - \xi_4 - \xi_5 + \xi_6 - 2\xi_7 - \xi_{10} - \xi_{11} \\ -\xi_1 - \xi_4 - \xi_5 - \xi_7 - \xi_{10} - \xi_{11} \\ \xi_1 + \xi_4 + \xi_5 + \xi_7 + \xi_{10} \\ -\xi_1 - \xi_2 - \xi_3 - \xi_4 - \xi_5 - \xi_6 - \xi_7 - \xi_8 - \xi_9 - \xi_{10} - \xi_{11} \end{pmatrix}. \tag{5.12}$$

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6. Partial resolutions

In order to see (partial) resolutions of the singularities in the formalism used above, we need to turn on the Fayet-Iliopoulos terms. This will correspond to triangulations of the convex cone in toric geometry or moving the center of the moment map in symplectic reduction.

Before starting the actual discussion, we make some observations about the general cases. In [14], the author considered the case of generalized conifolds of type $xy = u^k v^k$ and their resolutions. Their partial resolutions are conifold singularities, pinch point singularities and orbifold singularities and are obtained for different values of the FI parameters. In the T dual picture, D3 branes at $xy = u^k v^k$ singularities transform into k NS branes, k NS$'$ branes on circle together with D4 branes having the circle as one of the worldvolume coordinates. Partial resolutions of the singularity are obtained in the T-dual picture by moving one NS brane in the x^7 direction (in field theory this means to give expectation values to one field thus breaking the product of two gauge groups into a diagonal one). This smoothen the singularity to $xy = u^{k-1}v^k$. By removing a NS$'$ brane, the singularity is smoothen to $xy = u^k v^{k-1}$. In [14], the starting point was D3 at $xy = u^2 v^2$ singularity whose T dual contains 2 NS and 2 NS$'$ branes. By removing the two NS branes one arrives at the orbifold singularity, by removing one NS and one NS$'$ one arrives at the conifold and by removing either one NS or one NS$'$ the pinch point singularity is obtained. This of course means that we resolve the initial "worse" singularity to a "smoother" one. By removing NS branes we have complete control on the spacetime singularity.

In the case of orbifolded conifolds we need to use brane box models obtained by making two T-dualities. In this case, the resolutions are obtained either by moving NS and NS$'$ branes with respect to each other or by opening diamonds at the intersections of the NS and NS$'$ branes.

The discussion is similar for both types of $\mathbb{Z}_k \times \mathbb{Z}_l$ orbifolded conifolds discussed in this paper. Let us consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolded conifold case. From (4.12), we have the moment map $\mu_{\mathcal{M}} : \mathbb{C}^9 \to \mathbb{R}^6$:

$$\mu_{\mathcal{M}} = \begin{pmatrix} 2|p_3|^2 - 2|p_4|^2 - |p_6|^2 + |p_8|^2 - 2\xi_1 - \xi_2 - \xi_4 - \xi_5 + \xi_7 \\ |p_3|^2 - |p_4|^2 - |p_6|^2 + |p_7|^2 - \xi_1 + \xi_7 \\ |p_3|^2 - 2|p_4|^2 + |p_5|^2 - \xi_1 + \xi_3 + \xi_6 + \xi_7 \\ |p_0|^2 - 2|p_3|^2 + |p_6|^2 + \xi_1 + \xi_2 + \xi_5 + \xi_6 \\ |p_2|^2 - 2|p_4|^2 + |p_6|^2 + \xi_2 + \xi_3 + \xi_6 + \xi_7 \\ |p_1|^2 - |p_3|^2 - |p_4|^2 + |p_6|^2 + \xi_2 + \xi_6 \end{pmatrix} \tag{6.1}$$

where p_i are homogeneous coordinates of $\mathbb{C}^9$. Then $\mathcal{M}$ is the symplectic reduction $\mu_{\mathcal{M}}^{-1}(0)//\,\mathrm{U}(1)^6$. From the conditions of (4.13), the Fayet-Iliopoulos parameters of

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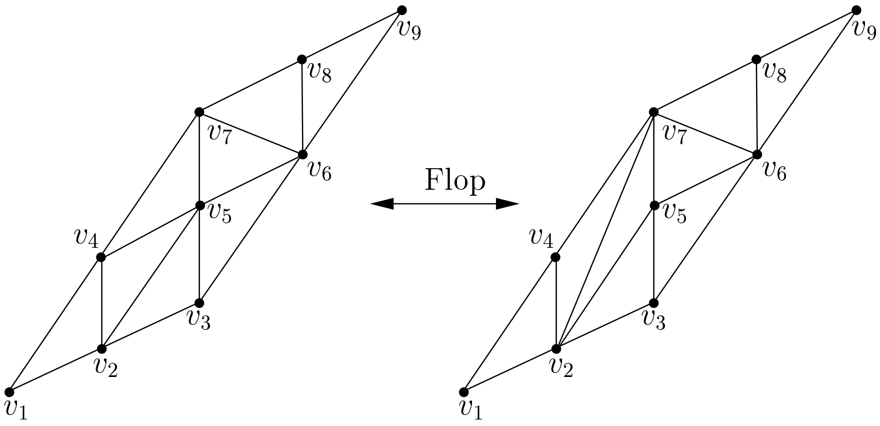

Figure 4: A flop transition between different phases.

the resulting $\mathrm{U}(1)^6$ gauged linear sigma model satisfy inequalities

$$\begin{aligned}-2\xi_1 - \xi_2 - \xi_4 - \xi_5 + \xi_7 &< 0\,,\\ -\xi_1 + \xi_7 &< 0\,,\\ -\xi_1 + \xi_3 + \xi_6 + \xi_7 &< 0\,,\\ \xi_1 + \xi_2 + \xi_5 + \xi_6 &< 0\,,\\ \xi_2 + \xi_3 + \xi_6 + \xi_7 &< 0\,.\end{aligned} \tag{6.2}$$

But the condition (4.13) does not determine the sign of the last coordinate $\xi_2 + \xi_6$ of the center of the moment map $\mu_{\mathcal{M}}$ in (6.1). The last coordinate of the moment map $\mu_{\mathcal{M}}$ gives us

$$|p_1|^2 - |p_3|^2 - |p_4|^2 + |p_6|^2 = -\xi_2 - \xi_6\,. \tag{6.3}$$

Thus when $\xi_2 + \xi_6 > 0$, it is parameterized by the homogeneous coordinates p_3 and p_4. When $\xi_2 + \xi_6 < 0$, it is parameterized by the homogeneous coordinates p_1 and p_6. This corresponds to two different triangulations of the rectangle $\{v_4, v_2, v_5, v_7\}$ in the toric diagram (see figure 4). These two phases are topologically different. Thus the D-brane vacuum moduli space $\mathcal{M}$ does have topologically distinct phases which are related by a flop transition. This phenomenon has been observed for orbifold singularities [22, 23]. We can see this flop in the toric diagram which is shown in figure 4 .

We now briefly discuss the relation between flops from the point of view of toric geometry and the one of brane configuration. For the case of generalized conifolds, there is no flop transition as observed in [14]. The T-duality picture for generalized conifolds is the one using perpendicular NS branes on an interval. By changing their positions one obtains a Seiberg duality but geometrically it is not a flop transition. In our case of orbifolded conifolds, the Seiberg duality for the

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geometrical Kähler transition are seen in the T-dual brane boxes where the resolution is given by the corresponding diamonds which open at the intersections of perpendicular NS branes. In [15], the Seiberg duality was described in terms of diamonds orientations. We conjecture here that going between two diamond configurations with different diamond orientation i.e. which are Seiberg dual will correspond to a flop in the geometry. The details of the calculations are left for a further work.

For special values of ξ_i, there are several singularity types. We get the orbifolded conifold $\mathcal{C}_{22}$ when all ξ are zero. But the singularity becomes partially resolved, when fields get expectation values in terms of the FI parameters. One of the most interesting case is when we give expectation values to the fields $A_{2i}, i = 1, \ldots, 4$. This region corresponds to $\xi_5 + \xi_1 = \xi_6 + \xi_2 = \xi_7 + \xi_3 = \xi_8 + \xi_4$. Hence the last three coordinates of the center of the moment map $\mu_{\mathcal{M}}$ are zeroes. Thus one can see that the lower left half triangle of the toric diagram will not be triangulated. So we will have an orbifold singularity $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_3$ for generic values of ξ_i under these circumstances (figure 5).

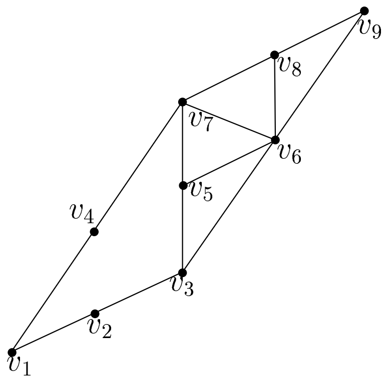

Figure 5: A phase with orbifold singularity.

The configuration of D3 branes at this singularity is T-dual to a 2×2 brane box with trivial identification of the unit cell. In the language of [15], giving expectation values to the fields A_{2i}, i.e. going to a baryonic branch, means to rotate the diamonds which lie at the intersections of the NS and NS' branes. One can have similar discussions for the $\mathbb{Z}_2 \times \mathbb{Z}_3$ orbifolded conifold.

7. Conclusions

In this paper, we have used the toric geometry and Witten's gauged linear sigma model approach to identify the Higgs moduli space of the field theory which exists on the world volume of D3 branes at a orbifolded conifold singularity of type $\mathcal{C}_{kl}$ which is a $\mathbb{Z}_k \times \mathbb{Z}_l$ quotient of the conifold $xy - uv = 0$. We have shown that the Higgs moduli space does have phases related by a flop transition and topology change can occur. We also observed that the orbifold singularity can be obtained as one of the phases of the Higgs moduli space. In field theory, this corresponds to giving expectation value to some hypermultiplets thus breaking parts of the gauge group.

Moreover, we have studied a correspondence between brane configurations involving D3 and perpendicular NS branes and brane at singularities for the case of orbifolded conifolds of type $\mathcal{C}_{kl}$.

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